

Pascal Formula for Binomial Coefficient

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The Pascal Formula for binomial coefficient is $(n > 1)$

$$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}, \quad \text{where} \quad C_r^n = \frac{n!}{r!(n-r)!}, \quad n! = n(n-1)\dots 3.2.1, \quad 0! = 1.$$

Proof 1

$$\begin{aligned} C_r^{n-1} + C_{r-1}^{n-1} &= \frac{(n-1)!}{r![(n-1)-r]!} + \frac{(n-1)!}{(r-1)![(n-1)-(r-1)]!} \quad \text{from definition} \\ &= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{(n-1)!}{r!(n-r)!} [(n-r)-r] \quad , \text{ take out HCF of numerators, LCM of denominators} \\ &= \frac{(n-1)!}{r!(n-r)!} [n] \\ &= \frac{n!}{r!(n-r)!} = C_r^n \end{aligned}$$

Proof 2

Binomial coefficient, C_r^n , equals the number of combinations of r items that can be selected from a set of n items, where the order of the r items taken out is unimportant.

Now, let us assume that in these n items, there is a special item, denoted by \odot .

We then have two cases :

(1) \odot is **not within** the r items taken out :

Since \odot is not taken out, we must choose from the $(n-1)$ remaining items r items out, giving the total number of combinations C_r^{n-1} .

(2) \odot is **within** the r items taken out :

Since \odot is taken out already, we need only to take out $(r-1)$ items from the $(n-1)$ remaining items, giving the total number of combinations C_{r-1}^{n-1} .

By adding the number of combinations given by (1) and (2) should give the total combination, that is, C_r^n .

Proof 3

Consider the identity : $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1} \dots (*)$

Coefficient of x^n -term in $(1+x)^n$ is C_r^n .

Coefficient of x^n -term in $(1+x)^{n-1}$ is C_r^{n-1} .

Coefficient of x^n -term in $x(1+x)^{n-1}$ is C_{r-1}^{n-1} .

Hence, from (*), we have : $C_r^n = C_r^{n-1} + C_{r-1}^{n-1}$.